

# A Two Parameter Family of the Calabi-Yau d-Fold

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## ABSTRACT

We study a two parameter family of Calabi-Yau d-fold by means of mirror symmetry. We construct mirror maps and calculate correlation functions associated with Kähler moduli in the original manifold. We find there are more complicated instanton corrections of these couplings than threefolds, which is expected to reflect families of instantons with continuous parameters.

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# 1 Introduction

It is a long time since the string theory attracted the attention as a candidate of the unified theory of elementary particles and their interactions. A lot of work has been devoted to the study of these theories, but it seems to be out of reach to gain fundamental understanding of them. One of the most important things is the investigation of the properties of the manifolds on which the string should be compactified. Particularly the compactifications on Calabi-Yau manifolds have received much attention [1, 2, 3]. From the point of view of the particle physics, cohomology classes of these Calabi-Yau manifolds correspond to zero-mass fields in the low energy effective theory and these manifolds play crucial roles in deciding the features of the string theories.

Originally Calabi-Yau threefolds were introduced to provide six dimensional inner space to yield a consistent string background. However it seemed hard to investigate their properties because of the quantum corrections [4, 5].

Recently a great progress has been achieved in understanding the properties of the moduli spaces of Calabi-Yau manifolds by the discovery of the mirror symmetry [6, 7, 8, 9, 10]. Now it has become possible to study the structure of the Kähler moduli space of the Calabi-Yau manifold to connect with the complex moduli space of its mirror manifold.

When one discusses the mirror symmetries, the (complex) dimension of the Calabi-Yau manifolds is restricted to be three because of the string theoretic applications [9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]. Nevertheless as the mathematical physics applications, generalized mirror manifold with other dimensions are of interest [22, 23, 24]. Lately one parameter family of the Calabi-Yau  $d$ -fold realized as a Fermat hypersurface embedded in a projective space of dimension  $(d+1)$  is studied [24]. The  $d$ -point correlation functions studied there have quantum corrections and these corrections are expected to correspond to Chern classes of various parameter spaces for rational curves on Calabi-Yau manifolds. This suggests the much richer structures in higher dimensional cases.

The aim of this paper is to construct a two parameter family of the mirror manifold paired with a Calabi-Yau  $d$ -fold and to investigate their properties in order to throw some new lights upon structures of the moduli spaces.

Firstly we construct the mirror manifold by the method of the toric varieties [25, 26]. Namely one constructs a mirror manifold as a zero locus of the Laurent polynomial in the ambient space using the information on the dual polyhedron  $\Delta^*$  [22, 27, 28, 29, 30]. The

deformations of the complex structures lead to the deformations of the Hodge dtructures. The information of the Hodge structures is encoded in periods of the manifold. One can correlate the Kähler moduli space of the original manifold with complex moduli space of its mirror manifold. Using the mirror maps, we identify the d point coupling of the complex moduli on the mirror manifold with that of the Kähler moduli on the original manifold. We can calculate quantum corrections of the latter with this correspondence.

The paper is organized as follows. In section 2, we construct a mirror maifold paired with a Calabi-Yau d-fold  $\mathbf{P}_{d+1}[2, 2, 2, \dots, 2, 2, 1, 1](2(d+1))$  in the toric language [22, 27, 28, 29, 30]. In section 3, we introduce Gel'fand, Kapranov and Zelevinsky equation system [31] and identify the Picard-Fuchs equations for the periods of the algebraic varieties. We construct mirror maps using solutions of this equation system. In section 4, the set of correlation functions associated with the complex moduli and the Kähler moduli are calculated. The former correlation functions are meromorphic functions with respect to the parameters of the manifolds and their singularities are characterized of the discriminant loci. By contrast, the latter correlation functions have quantum corrections. Also we study the Kähler cone of the original maifold. In section 5, monodromy matrices associated with the singular loci are obtained. Section 6 is devoted to conclusions and comments.

## 2 Construction of the mirror manifold

In this section, we consider a Calabi-Yau d-fold  $\mathcal{M}$  defined by a polynomial in the weighted projective space  $\mathbf{P}_{d+1}[2, 2, 2, \dots, 2, 2, 1, 1](2(d+1))$  and construct its mirror manifold  $\mathcal{W}$  in the recipe of the toric variety [25, 26, 22, 27, 28, 29, 30].

### 2.1 Polyhedron

We choose a lattice polyhedron  $\Delta$  to construct an ambient space. The Calabi-Yau manifold is realized by a hypersurface in this ambient space [25, 26, 28]. The lattice polyhedron  $\Delta$  is a convex hull  $d+1$  dimensinal rational polyhedron with vertices  $\nu_i$  ( $i = 1, 2, \dots, d+2$ ) in a lattice space  $M$ ,

$$\begin{aligned}\nu_1 &:= (d, -1, -1, -1, \dots, -1, -1) , \\ \nu_2 &:= (-1, d, -1, -1, \dots, -1, -1) ,\end{aligned}$$

$$\begin{aligned}
\nu_3 &:= (-1, -1, d, -1, \dots, -1, -1) , \\
\nu_4 &:= (-1, -1, -1, d, \dots, -1, -1) , \\
&\dots \\
\nu_d &:= (-1, -1, -1, -1, \dots, d, -1) , \\
\nu_{d+1} &:= (-1, -1, -1, -1, \dots, -1, 2d+1) , \\
\nu_{d+2} &:= (-1, -1, -1, -1, \dots, -1, -1) .
\end{aligned}$$

For each  $l$ -dimensional face  $\Xi_l \subset \Delta$ , a  $d+1$  dimensional cone  $\sigma(\Xi_l)$  is defined as,

$$\sigma(\Xi_l) := \{ \lambda \cdot (p - p') ; \lambda \in \mathbf{R}_+, p \in \Delta, p' \in \Xi_l \} .$$

We associate to  $\Delta$  a complete rational fan  $\Sigma(\Delta)$ , which is defined as the collection of  $(d+1-l)$  dimensional dual cones  $\sigma^*(\Xi_l)$  ( $l = 0, 1, \dots, d+1$ ). We use a toric variety  $\mathbf{P}_\Delta$  associated to this fan  $\Sigma(\Delta)$  as the ambient space in order to construct a Calabi-Yau manifold. The Calabi-Yau  $d$ -fold  $\mathcal{M} \equiv \mathcal{F}(\Delta)$  is realized as a zero locus  $Z_f$  of the Laurent polynomial  $f_\Delta$  in the algebraic torus  $(\mathbf{C}^*)^{d+1} \subset \mathbf{P}_\Delta$ ,

$$\begin{aligned}
f_\Delta(a, X) &:= \sum_{i=0}^s a_i X^{\nu_i} \\
&= a_0 \\
&\quad + a_1 X_1^d X_2^{-1} X_3^{-1} \cdots X_d^{-1} X_{d+1}^{-1} + a_2 X_1^{-1} X_2^d X_3^{-1} \cdots X_d^{-1} X_{d+1}^{-1} \\
&\quad + a_3 X_1^{-1} X_2^{-1} X_3^d \cdots X_d^{-1} X_{d+1}^{-1} + \cdots \\
&\quad + a_{d-1} X_1^{-1} X_2^{-1} \cdots X_{d-1}^d X_d^{-1} X_{d+1}^{-1} \\
&\quad + a_d X_1^{-1} X_2^{-1} \cdots X_{d-1}^{-1} X_d^d X_{d+1}^{-1} + a_{d+1} X_1^{-1} X_2^{-1} X_3^{-1} \cdots X_d^{-1} X_{d+1}^{2d+1} \\
&\quad + a_{d+2} X_1^{-1} X_2^{-1} X_3^{-1} \cdots X_d^{-1} X_{d+1}^{-1} + \cdots , \tag{1}
\end{aligned}$$

where the symbol “ $\cdots$ ” means that there are more monomial terms associated with the deformation of the complex structure. The toric variety  $\mathbf{P}_\Delta$  can be described,

$$\begin{aligned}
\mathbf{P}_\Delta &:= \{ [U_0, U_1, U_2, \dots, U_{d+1}, U_{d+2}] \in \mathbf{P}^{d+3} ; U_1^2 U_2^2 U_3^2 \cdots U_{d-1}^2 U_d^2 U_{d+1} U_{d+2} = U_0^{2(d+1)} \} , \\
U_1/U_0 &= X_1^d X_2^{-1} X_3^{-1} \cdots X_d^{-1} X_{d+1}^{-1} , \\
U_2/U_0 &= X_1^{-1} X_2^d X_3^{-1} \cdots X_d^{-1} X_{d+1}^{-1} , \\
U_3/U_0 &= X_1^{-1} X_2^{-1} X_3^d \cdots X_d^{-1} X_{d+1}^{-1} , \\
&\dots
\end{aligned}$$

$$\begin{aligned}
U_{d-1}/U_0 &= X_1^{-1}X_2^{-1}X_3^{-1} \cdots X_{d-1}^d X_d^{-1} X_{d+1}^{-1} , \\
U_d/U_0 &= X_1^{-1}X_2^{-1}X_3^{-1} \cdots X_{d-1}^{-1} X_d^d X_{d+1}^{-1} , \\
U_{d+1}/U_0 &= X_1^{-1}X_2^{-1}X_3^{-1} \cdots X_{d-1}^{-1} X_d^{-1} X_{d+1}^{2d+1} , \\
U_{d+2}/U_0 &= X_1^{-1}X_2^{-1}X_3^{-1} \cdots X_{d-1}^{-1} X_d^{-1} X_{d+1}^{-1} .
\end{aligned}$$

Considering the mapping,

$$X_1 = \frac{z_1}{z_{d+2}^2}, \quad X_2 = \frac{z_2}{z_{d+2}^2}, \quad X_3 = \frac{z_3}{z_{d+2}^2}, \quad \cdots, \quad X_d = \frac{z_d}{z_{d+2}^2}, \quad X_{d+1} = \frac{z_{d+1}}{z_{d+2}},$$

we can rewrite the zero locus  $Z_{f_\Delta}$

$$\begin{aligned}
&a_1 z_1^{d+1} + a_2 z_2^{d+1} + \cdots + a_{d-1} z_{d-1}^{d+1} + a_d z_d^{d+1} \\
&+ a_{d+1} z_{d+1}^{2(d+1)} + a_{d+2} z_{d+2}^{2(d+1)} + a_0 z_1 z_2 \cdots z_{d-1} z_d z_{d+1} z_{d+2} + \cdots = 0 .
\end{aligned}$$

This polynomial in the weighted projective space  $\mathbf{P}_{d+1}[2, 2, 2, \dots, 2, 2, 1, 1](2(d+1))$  determines a Calabi-Yau manifold  $\mathcal{F}(\Delta)$  [28].

## 2.2 Mirror manifold

A mirror variety  $\mathcal{W} \equiv \mathcal{F}(\Delta^*)$  of the manifold  $\mathcal{M} \equiv \mathcal{F}(\Delta)$  can be constructed by using a dual polyhedron  $\Delta^*$  of  $\Delta$  [28]. The dual polyhedron  $\Delta^*$  defined by

$$\Delta^* := \{(x_1, x_2, \dots, x_d, x_{d+1}) ; \sum_{i=1}^{d+1} x_i y_i \geq -1 \text{ for all } (y_1, y_2, \dots, y_d, y_{d+1}) \in \Delta\},$$

is again a rational polyhedron with vertices  $\nu_i^* \quad (i = 1, 2, \dots, d+2)$  in the lattice space  $N$ ,

$$\begin{aligned}
\nu_1^* &:= (1, 0, 0, \dots, 0, 0, 0) , \\
\nu_2^* &:= (0, 1, 0, \dots, 0, 0, 0) , \\
\nu_3^* &:= (0, 0, 1, \dots, 0, 0, 0) , \\
\nu_4^* &:= (0, 0, 0, \dots, 1, 0, 0) , \\
&\dots \\
\nu_d^* &:= (0, 0, 0, \dots, 0, 1, 0) , \\
\nu_{d+1}^* &:= (0, 0, 0, \dots, 0, 0, 1) , \\
\nu_{d+2}^* &:= (-2, -2, -2, \dots, -2, -2, -1) .
\end{aligned}$$

Both the polyhedron  $\Delta$  and the dual polyhedron  $\Delta^*$  are rational polyhedrons. Inside the  $\Delta^*$ , there are two integral points which describe the exceptional divisors [32, 33, 34, 35],

$$\begin{aligned}\nu_{d+3}^* &:= (-1, -1, -1, \dots, -1, -1, 0) , \\ \nu_0^* &:= (0, 0, 0, \dots, 0, 0, 0) .\end{aligned}$$

In the same way as in the previous subsection, we construct a Calabi-Yau variety  $\mathcal{W} \equiv \mathcal{F}(\Delta^*)$  associated to the polyhedron  $\Delta^*$ . The zero locus of the Laurent polynomial  $f_{\Delta^*}(b, Y)$  in the algebraic torus  $(\mathcal{C}^*)^{d+1} \subset \mathbf{P}_{\Delta^*}$  is defined,

$$\begin{aligned}f_{\Delta^*}(b, Y) &:= \sum_{i=0}^{d+3} b_i Y^{\nu_i^*} \\ &= b_0 + b_1 Y_1 + b_2 Y_2 + \dots + b_{d+2} Y_{d+2} + b_{d+3} Y_{d+3} .\end{aligned}\quad (2)$$

The toric variety  $\mathbf{P}_{\Delta^*}$  defined by,

$$\begin{aligned}\mathbf{P}_{\Delta^*} &:= \{ [V_0, V_1, V_2, \dots, V_{d+1}, V_{d+2}] \in \mathbf{P}^{d+3} ; V_1^2 V_2^2 V_3^2 \dots V_d^2 V_{d+1} V_{d+2} = V_0^{2(d+1)} \} , \\ V_1/V_0 &= Y_1 , V_2/V_0 = Y_2 , V_3/V_0 = Y_3 , \dots , V_{d-1}/V_0 = Y_{d-1} , V_d/V_0 = Y_d , \\ V_{d+1}/V_0 &= Y_{d+1} , V_{d+2}/V_0 = \frac{1}{Y_1^2 Y_2^2 \dots Y_{d-1}^2 Y_d^2 Y_{d+1}} , V_{d+3}/V_0 = \frac{1}{Y_1 Y_2 \dots Y_{d-1} Y_d} .\end{aligned}$$

is used for the ambient space. Let us introduce an etale map  $\varphi : \mathbf{P}^{d+1} \rightarrow \mathbf{P}_{\Delta}$  defined by,

$$\begin{aligned}\varphi &; [z_1, z_2, z_3, \dots, z_d, z_{d+1}, z_{d+2}] \\ &\rightarrow [z_1 z_2 z_3 \dots z_d z_{d+1} z_{d+2}, z_1^{d+1}, z_2^{d+1}, z_3^{d+1}, \dots, z_{d-1}^{d+1}, z_d^{d+1}, z_{d+1}^{2(d+1)}, z_{d+2}^{2(d+1)}] .\end{aligned}$$

Then the zero locus  $Z_{f_{\Delta^*}}$  can be rewritten,

$$\begin{aligned}b_1 z_1^{d+1} + b_2 z_2^{d+1} + \dots + b_{d-1} z_{d-1}^{d+1} + b_d z_d^{d+1} + b_{d+1} z_{d+1}^{2(d+1)} + b_{d+2} z_{d+2}^{2(d+1)} \\ + b_{d+3} z_{d+1}^{d+1} z_{d+2}^{d+1} + b_0 z_1 z_2 z_3 \dots z_d z_{d+1} z_{d+1} = 0 .\end{aligned}\quad (3)$$

The set  $(\Delta, M)$  is a reflexive pair and the set  $(\Delta^*, N)$  is a dual reflexive pair. The lattice polyhedron  $\Delta$  is a Fano polyhedron and determines the Gorenstein toric Fano variety  $\mathbf{P}_{\Delta}$  [25, 26, 28]. Similarly the dual polyhedron  $\Delta^*$  determines the dual Gorenstein toric Fano variety  $\mathbf{P}_{\Delta^*}$ . A zero locus  $Z_{f_{\Delta}}$  defined by a Laurent polyhedron  $f_{\Delta}$  with a fixed Newton polyhedron  $\Delta$  gives a Calabi-Yau variety  $\mathcal{F}(\Delta)$ . A Laurent polynomial  $f_{\Delta^*}$  with a fixed Newton polyhedron  $\Delta^*$  leads to another Calabi-Yau variety  $\mathcal{F}(\Delta^*)$ . Thus the mirror manifold  $\mathcal{F}(\Delta^*)$  of the  $\mathcal{F}(\Delta)$  is realized as the zero locus in the weighted projective space  $\mathbf{P}_{d+1}[2, 2, 2, \dots, 2, 2, 1, 1](2(d+1))$  [28]. In the following sections, we investigate various properties of this mirror manifold.

### 3 Picard-Fuchs equation and the mirror map

In this section, we introduce periods of the mirror manifold and write down a differential equation satisfied by the periods. We solve solutions of this equation and define the mirror maps of the Calabi-Yau d-fold as the ratios of the periods. These maps connect the complex structure of  $\mathcal{F}(\Delta^*)$  with the Kähler structure of  $\mathcal{F}(\Delta)$ .

#### 3.1 Periods

The structure of the moduli space of a Calabi-Yau manifold is described by giving the periods of the manifold. The period integrals of the  $\mathcal{F}(\Delta^*)$  are defined as [18, 27, 20, 21],

$$\Pi_i(b) := \int_{\Gamma_i} \frac{1}{f_{\Delta^*}(b, X)} \prod_{j=1}^{d+1} \frac{dX_j}{X_j} , \quad (4)$$

with the homology cycles  $\Gamma_i \in H_{d+1}((\mathbf{C}^*)^{d+1} \setminus Z_{f_{\Delta^*}}, \mathbf{Z})$ . If we choose the cycle  $\Gamma$ ,

$$\Gamma := \{ X_1, X_2, \dots, X_d, X_{d+1} \in \mathbf{C}^{d+1} ; |X_i| = 1 \ (i = 1, 2, \dots, d+1) \} ,$$

the integral is calculated explicitly and the fundamental period  $\Pi_0(b) = \frac{1}{b_0} \varpi_0(x, y)$  is obtained,

$$\varpi_0(x, y) = \sum_{n=0}^{\infty} \sum_{m \geq 2n}^{\infty} \frac{((d+1)m)!}{(m-2n)!(m!)^d(n!)^2} x^m y^n \quad (5)$$

$$x := \frac{b_1 b_2 b_3 \cdots b_{d-1} b_d b_{d+3}}{b_0^{d+1}}, \quad y := \frac{b_{d+1} b_{d+2}}{b_{d+3}^2} . \quad (6)$$

In the next subsection, we derive a differential equation satisfied by the periods. We solve this equation to determine the periods instead of carrying out the integral on the homology cycles explicitly.

#### 3.2 Generalized hypergeometric equation and the Picard-Fuchs equation

We will give the Picard-Fuchs equation satisfied by the period integral of the mirror manifold  $\mathcal{W} \equiv \mathcal{F}_{\Delta^*}$  in an efficient way [27]. Let us introduce the generalized hypergeometric system of Gel'fand, Kapranov and Zelevinsky [31] (GKZ equation) defined by a set of points  $\{\nu_0^*, \nu_1^*, \dots, \nu_{d+2}^*, \nu_{d+3}^*\}$  in the polyhedron  $\Delta^*$ . Points  $\nu_1^*, \nu_2^*, \dots, \nu_{d+1}^*, \nu_{d+2}^*$  lie at the vertices of  $\Delta^*$ . On the other hand, points  $\nu_{d+3}^*, \nu_0^*$  lie in the interior of codimension  $d$ , zero faces of  $\Delta^*$  respectively. We embed these integral points in  $\mathbf{R}^{d+2}$  by

$$\bar{\nu}_i^* := (1, \nu_i^*) \ (i = 0, 1, 2, \dots, d+2, d+3) .$$

Then these  $(d+4)$  points are linear dependent in  $\mathbf{R}^{d+2}$  and satisfy two linear relations,

$$\begin{aligned} -(d+1)\bar{\nu}_0^* + \bar{\nu}_1^* + \bar{\nu}_2^* + \cdots + \bar{\nu}_{d-1}^* + \bar{\nu}_d^* + \bar{\nu}_{d+3}^* &= 0 \ , \\ \bar{\nu}_{d+1}^* + \bar{\nu}_{d+2}^* - 2\bar{\nu}_{d+3}^* &= 0 \ . \end{aligned}$$

We associate two lattice vectors  $l^{(1)}, l^{(2)}$  in  $\mathbf{Z}^{d+4}$  to these linear relations,

$$l^{(1)} := (-(d+1), 1, 1, 1, \dots, 1, 1, 0, 0, 1) \ , \quad (7)$$

$$l^{(2)} := (0, 0, 0, 0, \dots, 0, 0, 1, 1, -2) \ . \quad (8)$$

Using the above things, we define differential operators,

$$\square_{l^{(i)}} := \prod_{l_j^{(i)} > 0} \left( \frac{\partial}{\partial b_j} \right)^{l_j^{(i)}} - \prod_{l_j^{(i)} < 0} \left( \frac{\partial}{\partial b_j} \right)^{-l_j^{(i)}} \ , \quad (9)$$

$$\begin{aligned} \mathcal{P} &:= \sum_{i=0}^{d+3} \bar{\nu}_i^* b_i \frac{\partial}{\partial b_i} - \beta \ , \\ \beta &:= (-1, 0, 0, 0, \dots, 0, 0, 0, 0, 0) \ . \end{aligned} \quad (10)$$

Then the GKZ differential equation system is defined [31, 20, 21],

$$\square_{l^{(1)}} \Phi = 0 \quad (11)$$

$$\square_{l^{(2)}} \Phi = 0 \quad (12)$$

$$\mathcal{P} \Phi = 0 \ . \quad (13)$$

Explicitly these can be re-expressed as ,

$$\left\{ \frac{\partial}{\partial b_1} \frac{\partial}{\partial b_2} \cdots \frac{\partial}{\partial b_{d-1}} \frac{\partial}{\partial b_d} \frac{\partial}{\partial b_{d+3}} - \left( \frac{\partial}{\partial b_1} \right)^{d+1} \right\} \Phi = 0 \ , \quad (14)$$

$$\left\{ \frac{\partial}{\partial b_{d+1}} \frac{\partial}{\partial b_{d+2}} - \left( \frac{\partial}{\partial b_{d+3}} \right)^2 \right\} \Phi = 0 \ , \quad (15)$$

and are also called  $\Delta^*$ -hypergeometric system together [31, 28] with the equation (13). In order to solve this system, we introduce the variables,

$$\begin{aligned} x &:= \frac{b_1 b_2 \cdots b_{d-1} b_d b_{d+3}}{b_0^{d+1}} \ , \quad y := \frac{b_{d+1} b_{d+2}}{b_{d+3}^2} \ , \\ \varpi(x, y) &:= b_0 \cdot \Phi(x, y) \ . \end{aligned}$$

Then the equation (13) is satisfied automatically, and the equations (14,15) are reduced to the following equation system,

$$\begin{aligned}\Theta_x \mathcal{D}_{l^{(1)}} \varpi(x, y) &= 0 , \\ \mathcal{D}_{l^{(2)}} \varpi(x, y) &= 0 , \\ \mathcal{D}_{l^{(1)}} &:= \Theta_x^{N-2} (\Theta_x - 2\Theta_y) \quad (16)\end{aligned}$$

$$-Nx(N\Theta_x + N - 1)(N\Theta_x + N - 2) \cdots (N\Theta_x + 2)(N\Theta_x + 1) , \quad (17)$$

$$\mathcal{D}_{l^{(2)}} := \Theta_y^2 - y(\Theta_x - 2\Theta_y)(\Theta_x - 2\Theta_y - 1) , \quad (18)$$

where  $\Theta_x := x \frac{d}{dx}$ ,  $\Theta_y := y \frac{d}{dy}$  and  $N := d + 1$ . In order to associate the solutions in the above system to the periods (4) defined in the section 3.1, we take the fundamental period  $\Pi_0 = \frac{1}{b_0} \varpi_0$ . As is seen easily,  $\varpi_0$  (5) satisfies an equation system,

$$\mathcal{D}_{l^{(1)}} \varpi_0(x, y) = 0 , \quad (19)$$

$$\mathcal{D}_{l^{(2)}} \varpi_0(x, y) = 0 . \quad (20)$$

Furthermore the periods  $\Pi_0$  satisfies the equation,

$$\mathcal{P} \Pi_0 = 0 . \quad (21)$$

Taking into account of this fact, we make the Ansatz that the period integrals  $\Pi(b)$  satisfy the above  $\Delta^*$ -hypergeometric system and are identified with the solutions  $\Phi(b)$  except that the  $b_0 \Pi(b)$  is included in the  $\text{Ker} \mathcal{D}_{l^{(1)}}$  rather than  $\text{Ker} \{\Theta_x \mathcal{D}_{l^{(1)}}\}$ . Under this Ansatz, we identify  $b_0 \Pi(b)$  with the solution  $\varpi(x, y)$  of the simultaneous differential equations,

$$\mathcal{D}_{l^{(1)}} \varpi(x, y) = 0 , \quad (22)$$

$$\mathcal{D}_{l^{(2)}} \varpi(x, y) = 0 . \quad (23)$$

and identify the Picard-Fuchs equations for the periods with the GKZ equation system.

### 3.3 The mirror map

In this section, we construct mirror maps  $t_1, t_2$  which connect the Kähler structure of the original manifold  $\mathcal{F}(\Delta)$  with the complex structure of the mirror manifold  $\mathcal{F}(\Delta^*)$  by using solutions of the  $\Delta^*$ -hypergeometric system.

The GKZ differential equation system has maximally unipotent monodromy [11] at a point  $(x, y) = (0, 0)$ . This point  $(x, y) = (0, 0)$  corresponds to the large complex structure of

the mirror manifold  $\mathcal{W} \equiv \mathcal{F}(\Delta^*)$ . We impose the condition that the large complex structure limit of the mirror manifold  $\mathcal{F}(\Delta)$  matches the large radius limit of the original manifold  $\mathcal{F}(\Delta)$ . In order to satisfy this claim, we take as a boundary condition of the mirror maps  $t_1, t_2$  the following conditions,

$$\begin{aligned} t_1(e^{2\pi i}x, y) &= t_1(x, y) + 1 , \\ t_2(x, e^{2\pi i}y) &= t_2(x, y) + 1 . \end{aligned}$$

These transformations are translations and correspond to the modular transformation at the infinity. These maps  $t_1, t_2$  can be expressed by the periods  $\varpi^{(0)}, \varpi^{(1)}, \varpi^{(2)}$ , which satisfy the GKZ equation system. Considering the boundary conditions, we can define the mirror maps  $t_1, t_2$ ,

$$t_1(x, y) := \frac{1}{2\pi i} \frac{\varpi^{(1)}}{\varpi^{(0)}} , \quad (24)$$

$$t_2(x, y) := \frac{1}{2\pi i} \frac{\varpi^{(2)}}{\varpi^{(0)}} . \quad (25)$$

In this formula,  $\varpi^{(0)}, \varpi^{(1)}, \varpi^{(2)}$  are solutions of the equations (22,23) and are given as,

$$\varpi^{(0)}(x, y) := \sum_{m, n \geq 0} \frac{\Gamma((d+1)m+1)}{[\Gamma(m+1)]^d} \frac{x^m y^n}{\Gamma(m-2n+1) [\Gamma(n+1)]^2} , \quad (26)$$

$$\begin{aligned} \varpi^{(1)}(x, y) &:= \varpi^{(0)}(x, y) \cdot \log x \\ &+ \sum_{m, n \geq 0} \frac{\Gamma((d+1)m+1)}{[\Gamma(m+1)]^d} \frac{x^m y^n}{\Gamma(m-2n+1) [\Gamma(n+1)]^2} \\ &\times [(d+1)\Psi((d+1)m+1) - d\Psi(m+1) - \Psi(m-2n+1)] , \end{aligned} \quad (27)$$

$$\begin{aligned} \varpi^{(2)}(x, y) &:= \varpi^{(0)}(x, y) \cdot \log y \\ &+ \sum_{m, n \geq 0} \frac{\Gamma((d+1)m+1)}{[\Gamma(m+1)]^d} \frac{x^m y^n}{\Gamma(m-2n+1) [\Gamma(n+1)]^2} \\ &\times 2[\Psi(m-2n+1) - \Psi(n+1)] . \end{aligned} \quad (28)$$

In the later convenience, we rewrite the maps  $t_1, t_2$  in the integral representation. Firstly we introduce the parameters  $\psi, \phi$ ,

$$\psi := -\frac{1}{2N} (x^2 y)^{-1/(2N)} , \quad \phi := -\frac{1}{2} y^{-1/2} .$$

Then the zero locus (3) becomes the following formula with a bit of rescaling of the set of variables,

$$\begin{aligned} p &= z_1^{d+1} + z_2^{d+1} + \cdots + z_{d-1}^{d+1} + z_d^{d+1} + z_{d+1}^{2(d+1)} + z_{d+2}^{2(d+1)} \\ &- 2(d+1)\psi z_1 z_2 \cdots z_{d-1} z_d z_{d+1} z_{d+2} - 2\phi z_{d+1}^{d+1} z_{d+2}^{d+1} = 0 . \end{aligned} \quad (29)$$

With this new variables, the series (26) can be rewritten as

$$\begin{aligned}\varpi^{(0)} &:= \varpi^{(0)}(\psi, \phi) \\ &= \sum_{n=0}^{\infty} \frac{\Gamma((d+1)n+1)}{[\Gamma(n+1)]^{d+1}} \frac{(-1)^n}{(2(d+1)\psi)^{(d+1)n}} u_n(\phi) ,\end{aligned}\quad (30)$$

$$\begin{aligned}u_{\nu}(\phi) &:= (2\phi)^{\nu} {}_2F_1\left(-\frac{\nu}{2}, -\frac{\nu}{2} + \frac{1}{2}, 1; \frac{1}{\phi^2}\right) . \\ \text{for } \left|\frac{2^d \psi^{d+1}}{\phi \pm 1}\right| &> 1 \text{ and } -\pi < \arg\left(\frac{2^d \psi^{d+1}}{\phi \pm 1}\right) < \pi .\end{aligned}\quad (31)$$

By the analytic continuation in the region

$$\left|\frac{2^d \psi^{d+1}}{\phi \pm 1}\right| < 1 ,$$

$\varpi_0(\psi, \phi)$  can be expressed,

$$\varpi_0(\psi, \phi) = -\frac{1}{(d+1)\pi^d} \sum_{m=1}^{\infty} \frac{\left[\Gamma\left(\frac{m}{d+1}\right)\right]^{d+1}}{\Gamma(m)} \times \left(\sin \frac{\pi m}{d+1}\right)^d (-2(d+1)\psi)^m u_{-\frac{m}{d+1}}(\phi) . \quad (32)$$

A complete set of solutions valid in this range is easily constructed. The set of solutions  $\{\varpi_j\}$  ( $j = 0, 1, 2, \dots, d+3, d+4$ ) defined by,

$$\begin{aligned}\varpi_j(\psi, \phi) &= \varpi_0(\alpha^j \psi, (-1)^j \phi) \\ &= -\frac{1}{(d+1)\pi^d} \sum_{m=1}^{\infty} \frac{\left[\Gamma\left(\frac{m}{d+1}\right)\right]^{d+1}}{\Gamma(m)} \\ &\quad \times \left(\sin \frac{\pi m}{d+1}\right)^d (-2(d+1)\psi)^m \alpha^{jm} u_{-\frac{m}{d+1}}((-1)^j \phi) ,\end{aligned}\quad (33)$$

$$\alpha = \exp \frac{2\pi i}{2(d+1)} , \quad (34)$$

expands the complete set of solutions. Integral representations of these solution can be represented as,

$$\varpi_{2j}(\psi, \phi) = \frac{-1}{(d+1)\pi^d} \sum_{r=1}^d (-1)^r \left(\sin \frac{\pi r}{d+1}\right)^d \cdot \alpha^{2jr} \xi_r(\psi, \phi) , \quad (35)$$

$$\varpi_{2j+1}(\psi, \phi) = \frac{-1}{(d+1)\pi^d} \sum_{r=1}^d (-1)^r \left(\sin \frac{\pi r}{d+1}\right)^d \cdot \alpha^{(2j+1)r} \eta_r(\psi, \phi) , \quad (36)$$

$$\xi_r(\psi, \phi) = \int_C \frac{d\nu}{2i \sin \pi \left(\nu + \frac{r}{d+1}\right)} \cdot \frac{[\Gamma(-\nu)]^{d+1}}{\Gamma(-(d+1)\nu)} \cdot (2(d+1)\psi)^{-(d+1)\nu} u_{\nu}(\phi) , \quad (37)$$

$$\begin{aligned}\eta_r(\psi, \phi) &= \int_C \frac{-d\nu}{2i \sin \pi \left(\nu + \frac{r}{d+1}\right)} \cdot \frac{[\Gamma(-\nu)]^{d+1}}{\Gamma(-(d+1)\nu)} \cdot (2(d+1)\psi)^{-(d+1)\nu} \\ &\quad \times \frac{u_{\nu}(\phi) \sin \pi \left(\nu + \frac{r}{d+1}\right) + u_{\nu}(-\phi) \sin \frac{\pi r}{d+1}}{\sin \pi \nu} , \quad (j = 0, 1, 2, \dots, d-1) ,\end{aligned}\quad (38)$$

where the contour  $C$  is chosen to enclose the poles  $\nu = -n - \frac{r}{d+1}$  ( $n = 0, 1, 2, \dots$ ) counterclockwise. With this basis, the mirror maps  $t_1, t_2$  (24,25) can be re-expressed,

$$2\pi i t_1 = \frac{-\varpi_0 + (d-1)\varpi_2 + (d-2)\varpi_4 + \dots + 2\varpi_{2(d-2)} + \varpi_{2(d-1)}}{(d+1)\varpi_0} , \quad (39)$$

$$= -\frac{1}{2} \quad (40)$$

$$+ \frac{1}{4\varpi_0} \int_C \frac{d\nu}{(\sin \pi\nu)^2} \cdot \frac{\Gamma((d+1)\nu+1)}{[\Gamma(\nu+1)]^{d+1}} \cdot (2(d+1)\psi)^{-(d+1)\nu} u_\nu(\phi) \cos \pi\nu , \quad (41)$$

$$2\pi i t_2 = -\frac{-\varpi_0 + (d-1)\varpi_2 + (d-2)\varpi_4 + \dots + 2\varpi_{2(d-2)} + \varpi_{2(d-1)}}{(d+1)\varpi_0} + \frac{d\varpi_1 + (d-1)\varpi_3 + (d-2)\varpi_5 + \dots + 2\varpi_{2d-3} + \varpi_{2d-1}}{(d+1)\varpi_0} , \quad (42)$$

$$= \frac{1}{2} - \frac{1}{4\varpi_0} \int_C \frac{d\nu}{(\sin \pi\nu)^2} \cdot \frac{\Gamma((d+1)\nu+1)}{[\Gamma(\nu+1)]^{d+1}} \cdot (2(d+1)\psi)^{-(d+1)\nu} \times [u_\nu(\phi) \cos \pi\nu - u_\nu(-\phi)] . \quad (43)$$

When we take the contour  $C$  to enclose the poles  $\nu = -n - \frac{r}{d+1}$  ( $n = 0, 1, 2, \dots$ ) counterclockwise, the above formulae coincide with the representation (24,25).

## 4 The correlation functions

In this section, we calculate d-point correlation functions associated with the complex structure of mirror manifold  $\mathcal{F}(\Delta^*)$ . Also we investigate the Kähler cone of the original manifold  $\mathcal{F}(\Delta)$  and d-point correlation functions of cohomology classes associated with the Kähler structure.

### 4.1 Correlation functions associated with the complex structure

It is known that there exists a nowhere-vanishing holomorphic  $d$ -form  $\Omega$  with respect to a canonical basis  $(\alpha_i, \beta^j) \in H^d((\mathbf{C}^*)^{d+1} \setminus Z_{f_{\Delta^*}}, \mathbf{Z})$

$$\Omega = \sum_i (z^i \alpha_i - g_i \beta^i)$$

with

$$\begin{aligned} \int_{\mathcal{W}} \alpha_i \wedge \alpha_j &= \int_{\mathcal{W}} \beta^i \wedge \beta^j = 0 , \\ \int_{\mathcal{W}} \alpha_i \wedge \beta^j &= \delta_i^j , \end{aligned}$$

The set  $(z^i, g_j)$  are realized as linear combinations of the periods and can be written as linear combination of the solutions of the GKZ equation (22,23).

Now let us consider a change of the complex structure of the Calabi-Yau manifold. With the help of the Kodaira Spencer theorem, the following relations are understood,

$$\begin{aligned}
\frac{\partial \Omega}{\partial \zeta^{i_1}} &\in H^{(d,0)} \otimes H^{(d-1,1)} , \\
\frac{\partial^2 \Omega}{\partial \zeta^{i_1} \partial \zeta^{i_2}} &\in H^{(d,0)} \otimes H^{(d-1,1)} \otimes H^{(d-2,2)} , \\
\frac{\partial^3 \Omega}{\partial \zeta^{i_1} \partial \zeta^{i_2} \partial \zeta^{i_3}} &\in H^{(d,0)} \otimes H^{(d-1,1)} \otimes H^{(d-2,2)} \otimes H^{(d-3,3)} , \\
&\cdots \\
\frac{\partial^{d-1} \Omega}{\partial \zeta^{i_1} \partial \zeta^{i_2} \cdots \partial \zeta^{i_{d-1}}} &\in H^{(d,0)} \otimes H^{(d-1,1)} \otimes H^{(d-2,2)} \otimes \cdots \otimes H^{(2,d-2)} \otimes H^{(1,d-1)} , \\
\frac{\partial^d \Omega}{\partial \zeta^{i_1} \partial \zeta^{i_2} \cdots \partial \zeta^{i_{d-1}} \partial \zeta^{i_d}} &\in H^{(d,0)} \otimes H^{(d-1,1)} \otimes H^{(d-2,2)} \otimes \cdots \otimes H^{(2,d-2)} \otimes H^{(1,d-1)} \otimes H^{(0,d)} .
\end{aligned}$$

Clearly the following equations are satisfied,

$$\begin{aligned}
\int_{\mathcal{W}} \Omega \wedge \frac{\partial \Omega}{\partial \zeta^{i_1}} &= 0 , \quad \int_{\mathcal{W}} \Omega \wedge \frac{\partial^2 \Omega}{\partial \zeta^{i_1} \partial \zeta^{i_2}} = 0 , \\
\int_{\mathcal{W}} \Omega \wedge \frac{\partial^3 \Omega}{\partial \zeta^{i_1} \partial \zeta^{i_2} \partial \zeta^{i_3}} &= 0 , \quad \cdots \quad \int_{\mathcal{W}} \Omega \wedge \frac{\partial^{d-1} \Omega}{\partial \zeta^{i_1} \partial \zeta^{i_2} \cdots \partial \zeta^{i_{d-2}} \partial \zeta^{i_{d-1}}} = 0 . \quad (44)
\end{aligned}$$

We can define d-point correlation functions  $\kappa_{i_1 i_2 \cdots i_d}$ ,

$$\kappa_{i_1 i_2 \cdots i_d} := - \int_{\mathcal{W}} \Omega \wedge \frac{\partial^d \Omega}{\partial \zeta^{i_1} \partial \zeta^{i_2} \partial \zeta^{i_3} \cdots \partial \zeta^{i_{d-1}} \partial \zeta^{i_d}} \quad (45)$$

$$= \sum_n (z^n \partial_{i_1} \partial_{i_2} \cdots \partial_{i_{d-1}} \partial_{i_d} g_n - g_n \partial_{i_1} \partial_{i_2} \cdots \partial_{i_{d-1}} \partial_{i_d} z^n) . \quad (46)$$

Because the sets  $(z^n, g_n)$  can be expressed as linear combinations of the solutions  $\varpi(x, y)$  of the GKZ equation, we can obtain the couplings  $\kappa_{i_1 i_2 \cdots i_d}$  from the information on the GKZ equation. For convenience, we define the following variables,

$$\bar{x} := N^N x , \quad \bar{y} := 4y , \quad (47)$$

$$W_{a,b} := \sum_n (z^n \partial_{\bar{x}}^a \partial_{\bar{y}}^b g_n - g_n \partial_{\bar{x}}^a \partial_{\bar{y}}^b z^n) , \quad (48)$$

Then the differential operators  $\mathcal{D}_{l^{(1)}}, \mathcal{D}_{l^{(2)}}$  (17,18) can be rewritten as,

$$\begin{aligned}
\mathcal{D}_{l^{(1)}} &= \bar{x}^{N-1} (1 - \bar{x}) \partial_{\bar{x}}^{N-1} - 2 \bar{x}^{N-2} \bar{y} \partial_{\bar{x}}^{N-2} \partial_{\bar{y}} + \bar{x}^{N-2} \left( \frac{(N-1)(N-2)}{2} - \frac{(N-1)^2}{2} \bar{x} \right) \partial_{\bar{x}}^{N-2} \\
&\quad - (N-2)(N-3) \bar{x}^{N-3} \bar{y} \partial_{\bar{x}}^{N-3} \partial_{\bar{y}} + \cdots , \\
\mathcal{D}_{l^{(2)}} &= \bar{y}^2 (1 - \bar{y}) \partial_{\bar{y}}^2 + \bar{y} \left( 1 - \frac{3}{2} \bar{y} \right) \partial_{\bar{y}} + \bar{x} \bar{y}^2 \partial_{\bar{x}} \partial_{\bar{y}} - \frac{1}{4} \bar{x}^2 \bar{y} \partial_{\bar{x}}^2 ,
\end{aligned}$$

and the GKZ equation system is written,

$$\mathcal{D}_{l^{(1)}}\varpi = 0, \quad \mathcal{D}_{l^{(2)}}\varpi = 0. \quad (49)$$

The sets  $(z^n, g_n)$  are linear combinations of the above solutions  $\varpi$ . The  $W_{a,b}$  satisfy the following equations,

$$\begin{aligned} \bar{x}(1-\bar{x})W_{d,0} - 2\bar{y}W_{d-1,1} &= 0, \\ \bar{y}(1-\bar{y})W_{0,d} + \bar{x}\bar{y}W_{1,d-1} - \frac{1}{4}\bar{x}^2W_{2,d-2} &= 0, \\ \bar{y}(1-\bar{y})W_{1,d-1} + \bar{x}\bar{y}W_{2,d-2} - \frac{1}{4}\bar{x}^2W_{3,d-3} &= 0, \\ \bar{y}(1-\bar{y})W_{2,d-2} + \bar{x}\bar{y}W_{3,d-3} - \frac{1}{4}\bar{x}^2W_{d-1,1} &= 0, \\ \dots \\ \bar{y}(1-\bar{y})W_{d-2,2} + \bar{x}\bar{y}W_{d-1,1} - \frac{1}{4}\bar{x}^2W_{d,0} &= 0. \end{aligned}$$

Using several identities derived from (44),

$$\begin{aligned} W_{N,0} &= \frac{N}{2}\partial_{\bar{x}}W_{5,0}, \\ W_{N-1,1} &= \frac{1}{2}((N-1)\partial_{\bar{x}}W_{N-2,1} + \partial_{\bar{y}}W_{N-1,0}), \\ W_{N-2,2} &= \frac{1}{2}((N-2)\partial_{\bar{x}}W_{N-3,2} + 2\partial_{\bar{y}}W_{N-2,1}), \end{aligned}$$

we calculate the  $d$  point coupling  $W_{a,b}$  ( $a+b=d$ ) up to a common overall constant factor by solving the simultaneous equation,

$$\begin{aligned} \bar{x}^2(1-\bar{x})W_{N,0} - 2\bar{x}\bar{y}W_{N-1,1} + \bar{x}\left(\frac{N(N-1)}{2} - \frac{N^2+1}{2}\bar{x}\right)W_{N-1,0} \\ -(N-1)(N-2)\bar{y}W_{N-2,1} = 0, \\ \bar{y}(1-\bar{y})W_{N-2,2} + \left(1 + \left(N - \frac{7}{2}\right)\bar{y}\right)W_{N-2,1} - \frac{N-2}{2}\bar{x}W_{N-1,0} \\ + \bar{x}\bar{y}W_{N-1,1} - \frac{1}{4}\bar{x}^2W_{N,0} = 0. \end{aligned}$$

Thus we obtain the solutions,

$$\begin{aligned} W_{d,0} &= \frac{1}{\Delta_1\Delta_3^d}, \\ W_{d-1,1} &= \frac{1-\bar{x}}{2\Delta_1\Delta_3^{d-1}\Delta_4}, \\ W_{d-2,2} &= \frac{-1+2\bar{x}}{4\Delta_1\Delta_2\Delta_3^{d-2}\Delta_4}, \end{aligned}$$

$$\begin{aligned}
W_{d-3,3} &= \frac{1 - \bar{x} + \bar{y} - 3\bar{x}\bar{y}}{8\Delta_1\Delta_2^2\Delta_3^{d-3}\Delta_4^2} , \\
W_{d-4,4} &= \frac{-3 + 4\bar{x} - \bar{y} + 4\bar{x}\bar{y}}{16\Delta_1\Delta_2^3\Delta_3^{d-4}\Delta_4^2} , \\
W_{d-5,5} &= \frac{1 - \bar{x} + 6\bar{y} - 10\bar{x}\bar{y} + \bar{y}^2 - 5\bar{x}\bar{y}^2}{32\Delta_1\Delta_2^4\Delta_3^{d-5}\Delta_4^3} .
\end{aligned}$$

We obtain general formula ,

$$W_{d-l,l} = \frac{f_l(\bar{x}, \bar{y})}{2^l \Delta_1 \Delta_2^{l-1} \Delta_3^{d-l} \Delta_4^{\lceil \frac{l+1}{2} \rceil}} , \quad (l = 1, 2, \dots, d) .$$

where

$$\begin{aligned}
f_{2m+1} &= \frac{1}{2} \sum_{l=0}^{m-1} {}_{2m}C_{2l+1} \cdot \bar{y}^l \cdot f_3 - (1 - \bar{y})^2 \sum_{l=0}^{m-2} {}_{2(m-1)}C_{2l+1} \cdot \bar{y}^l \cdot f_1 , \\
f_{2m+2} &= \frac{1}{2} \sum_{l=0}^{m-1} {}_{2m}C_{2l+1} \cdot \bar{y}^l \cdot f_4 - (1 - \bar{y})^2 \sum_{l=0}^{m-2} {}_{2(m-1)}C_{2l+1} \cdot \bar{y}^l \cdot f_2 , \\
f_1 &:= 1 - \bar{x} , \quad f_2 := -1 + 2\bar{x} , \\
f_3 &:= 1 - \bar{x} + \bar{y} - 3\bar{x}\bar{y} , \quad f_4 := -3 + 4\bar{x} - \bar{y} + 4\bar{x}\bar{y} , \\
&\quad (m = 2, 3, 4, \dots) .
\end{aligned}$$

Here the  $\Delta_1, \Delta_2, \Delta_3, \Delta_4$  are discriminants of the differential equation and are defined as,

$$\begin{aligned}
\Delta_1 &= (1 - \bar{x})^2 - \bar{x}^2 \bar{y} , \\
\Delta_2 &= 1 - \bar{y} , \\
\Delta_3 &= \bar{x} , \\
\Delta_4 &= \bar{y} .
\end{aligned}$$

The above couplings get singular when these discriminant loci vanish. In short, the singular properties of the correlation function associated with the complex structure of the  $\mathcal{F}(\Delta^*)$  are encoded in the discriminant loci of the  $\Delta^*$ -hypergeometric system. Using this formula, we can calculate Kähler couplings  $K_{a,b}$ .

$$K_{a,b} := \frac{1}{\varpi_0^2} \sum_{l+m=a, m+r=b, l+m=c, n+r=d} W_{c,d} \cdot \left( \frac{dx}{dt_1} \right)^l \cdot \left( \frac{dx}{dt_2} \right)^m \cdot \left( \frac{dy}{dt_1} \right)^n \cdot \left( \frac{dy}{dt_2} \right)^r {}_aC_l \cdot {}_bC_m .$$

We obtain the following results up to degree one,

$$K_{d,0} := 2N + 2N \cdot \left\{ 2 \cdot N^N - (N+1) \cdot N! - (N-1) \cdot N! \left( \sum_{m=2}^N \frac{N}{m} \right) \right\} q_1 + \dots ,$$

$$K_{d,0} := N + N \cdot \left\{ N^N - 2 \cdot N! - (N-2) \cdot N! \left( \sum_{m=2}^N \frac{N}{m} \right) \right\} q_1 + \cdots ,$$

$$K_{d,0} := 0 + \cdots ,$$

$$K_{d,0} := N q_2 + \cdots ,$$

$$K_{d-n,n} := 0 + \cdots , \quad (n \geq 4) ,$$

$$q_1 := \exp(2\pi i t_1) \quad q_2 := \exp(2\pi i t_2) \quad .$$

## 4.2 The Kähler cone

In this subsection, we will be concerned with the Kähler moduli space of the original manifold  $\mathcal{F}(\Delta)$ . This space has the structure of a cone, called the Kähler cone in contrast with the structure of the complex complex moduli space of the  $\mathcal{F}(\Delta^*)$ . The Kähler cone is defined by the requirement of a Kähler form on the Calabi-Yau manifold  $\mathcal{F}(\Delta)$ ,

$$\int_{\mathcal{M}} K \wedge K \wedge K \wedge \cdots \wedge K \wedge K > 0 , \quad \int_{S(d-1)} K \wedge K \wedge \cdots \wedge K \wedge K > 0 ,$$

$$\int_{S(d-2)} K \wedge K \wedge \cdots \wedge K > 0 , \cdots , \int_{S(2)} K \wedge K > 0 , \quad \int_{S(1)} K > 0 ,$$

where  $S(l)$  ( $l = 1, 2, \dots, d-1$ ) means the  $l$ -dimensional homologically nontrivial hypersurfaces in  $\mathcal{F}(\Delta)$ . The above requirement are translated into the toric language [32, 20, 21]. Firstly let us consider the new polyhedron  $\bar{\Delta}^* := (1, \Delta^*)$  in  $\mathbf{R}^{d+2}$ . We decompose the polyhedron  $\bar{\Delta}^*$  into simplices. When we take a  $(d+2)$  dimensional simplex  $\sigma$ , an arbitrary point  $v \in \sigma$  can be written as,

$$v = c_{i_0} \bar{\nu}_{i_0}^* + c_{i_1} \bar{\nu}_{i_1}^* + \cdots + c_{i_N} \bar{\nu}_{i_N}^* ,$$

$$\text{with } c_{i_0} + c_{i_1} + \cdots + c_{i_N} \leq 1 , \quad c_{i_0} > 0 , \quad c_{i_1} > 0 , \cdots , \quad c_{i_N} > 0 ,$$

where  $\bar{\nu}_{i_0}^*, \bar{\nu}_{i_1}^*, \dots, \bar{\nu}_{i_N}^*$  are the vertices of the  $\sigma$  and the subset of the  $\{\bar{\nu}_0^*, \bar{\nu}_1^*, \dots, \bar{\nu}_{N+2}^*\}$ . A piecewise linear function  $u$  is determined by giving a real value  $u_i \in \mathbf{R}$  for each vertex  $\bar{\nu}_i^* \in \bar{\Delta}^*$ . Then the piecewise linear function  $u$  takes the value for the point  $v \in \sigma$ ,

$$u(v) = c_{i_0} u_{i_0} + c_{i_1} u_{i_1} + \cdots + c_{i_N} u_{i_N} .$$

Equivalently, one can define this piecewise linear function by associating a vector  $z_\sigma \in \mathbf{R}^{N+1}$  to each  $(d+2)$  dimensional simplex  $\sigma$ ,

$$u(v) := \langle z_\sigma, v \rangle \text{ for all } v \in \sigma ,$$

where  $\langle *, * \rangle$  is the Euclidean inner product. Out of the piecewise linear functions, strictly convex piecewise linear functions defined by,

$$\begin{aligned} u(v) &= \langle z_\sigma, v \rangle \text{ when } v \in \sigma , \\ u(v) &> \langle z_\sigma, v \rangle \text{ when } v \notin \sigma , \end{aligned}$$

play an important role. The strictly convex piecewise linear functions constitute a cone in the following quotient space  $V$ ,

$$V := \left\{ \sum_{i=0}^{d+3} \mathbf{R} e_{\bar{\nu}_i^*} \right\} / \sim ,$$

where the symbol " $\sim$ " means the relations between the vectors  $e_{\bar{\nu}_i^*}$ ,

$$\sim ; \sum_{i=0}^{d+3} \langle x, \bar{\nu}_i^* \rangle e_{\bar{\nu}_i^*} = 0 \text{ for all } x \in \mathbf{R}^{d+1} . \quad (50)$$

In the context of the toric variety, this cone can be identified with the Kähler cone of  $\mathbf{P}_\Delta$  [25, 28, 29, 20]. By means of this method, we can get the Kähler cone of  $\mathcal{F}(\Delta)$ . If we take a  $(d+2)$  dimensional simplex  $\sigma$ ,

$$\sigma = \langle 0, \bar{\nu}_0^*, \bar{\nu}_2^*, \bar{\nu}_3^*, \bar{\nu}_4^*, \dots, \bar{\nu}_{d+1}^*, \bar{\nu}_{d+2}^* \rangle ,$$

then the vector  $z_\sigma$  can be written,

$$\begin{aligned} z_\sigma &= (u_0, du_0 - u_2 - \dots - u_{d-1} - u_d - \frac{1}{2}u_{d+1} - \frac{1}{2}u_{d+2}, \\ &\quad -u_0 + u_2, -u_0 + u_3, -u_0 + u_4, \dots, -u_0 + u_{d+1}) . \end{aligned}$$

The conditions of the strict convexity on  $u$  reads

$$\begin{aligned} -2(d+1)u_0 + 2(u_1 + u_2 + \dots + u_{d-1} + u_d) + u_{d+1} + u_{d+2} &> 0 , \\ 2u_{d+3} - u_{d+1} - u_{d+2} &> 0 . \end{aligned}$$

Also the relations (50) are written down explicitly,

$$\begin{aligned} e_{\bar{\nu}_0^*} + e_{\bar{\nu}_1^*} + e_{\bar{\nu}_2^*} + \dots + e_{\bar{\nu}_{d-1}^*} + e_{\bar{\nu}_d^*} + e_{\bar{\nu}_{d+1}^*} + e_{\bar{\nu}_{d+2}^*} + e_{\bar{\nu}_{d+3}^*} &= 0 , \\ e_{\bar{\nu}_1^*} = e_{\bar{\nu}_2^*} = \dots = e_{\bar{\nu}_{d-1}^*} = e_{\bar{\nu}_d^*} &= 2e_{\bar{\nu}_{d+2}^*} + e_{\bar{\nu}_{d+3}^*} , \\ e_{\bar{\nu}_{d+1}^*} &= e_{\bar{\nu}_{d+2}^*} . \end{aligned}$$

Then the generic element  $K_u$  in  $V$  can be expressed as,

$$\begin{aligned}
K_u &= - \sum_{i=0}^{d+3} u_i e_{\bar{\nu}_i^*} \\
&= \frac{1}{2(d+1)} [-2(d+1)u_0 + 2(u_1 + u_2 + \cdots + u_{d-1} + u_d) + u_{d+1} + u_{d+2}] e_{\bar{\nu}_0^*} \\
&\quad + \frac{1}{2} [u_{d+1} + u_{d+2} - 2u_{d+3}] e_{\bar{\nu}_{d+3}^*} \\
&= \frac{1}{2(d+1)} \langle u, 2l^{(1)} + l^{(2)} \rangle e_{\bar{\nu}_0^*} + \frac{1}{2} \langle u, l^{(2)} \rangle e_{\bar{\nu}_{d+3}^*} ,
\end{aligned} \tag{51}$$

with the conditions,

$$\langle u, 2l^{(1)} + l^{(2)} \rangle > 0, \quad \langle u, l^{(2)} \rangle < 0 .$$

In the above formulae,  $l^{(1)}, l^{(2)}$  are vectors defined in (7,8) in section 3.2. Under the identification of the bases  $e_{\bar{\nu}_0^*}, e_{\bar{\nu}_{d+3}^*}$  with divisors of  $\mathbf{P}_\Delta$ , the  $K_u$  can be interpreted as the Kähler cone in this model.

### 4.3 Correlation functions associated with the Kähler moduli

D point correlation functions associated with the Kähler class of the manifold  $\mathcal{F}(\Delta)$  can be derived by using the mirror maps and the correlation functions  $W_{a,b}$  ( $a+b=d$ ) given in section 4.1. These correlation functions on the Kähler moduli space have quantum corrections, which are expected to be interpreted in the geometrical terms. Firstly we want to find the variable  $\tilde{t}_i$  ( $i=1, 2$ ) associated with the integral basis  $h_i \in H^{1,1}(\mathcal{M}, \mathbf{Z})$ . With this basis, the Kähler form are expanded,

$$K_u = \tilde{t}_1 h_1 + \tilde{t}_2 h_2 .$$

From the results in the previous subsection, the Kähler cone  $K_u$  is also written,

$$K_u = \frac{1}{2(d+1)} \langle u, 2l^{(1)} + l^{(2)} \rangle e_{\bar{\nu}_0^*} + \frac{1}{2} \langle u, l^{(2)} \rangle e_{\bar{\nu}_{d+3}^*} .$$

Recall the asymptotic behaviour of the mirror maps in the large radius limit,

$$\begin{aligned}
t_1 &\sim \frac{1}{2\pi i} \log x = \frac{1}{2\pi i} \log \frac{b_1 b_2 \cdots b_{d-1} b_d b_{d+3}}{b_0^{d+1}} , \\
t_2 &\sim \frac{1}{2\pi i} \log y = \frac{1}{2\pi i} \log \frac{b_{d+1} b_{d+2}}{b_{d+3}^2} .
\end{aligned}$$

We impose the asymptotic relations of the piecewise linear function  $u$ ,

$$u_i = \log b_i \quad (i = 0, 1, 2, \dots, d+2, d+3) .$$

With this Ansatz, one can rewrite the Kähler cone  $K_u$ ,

$$\begin{aligned} K_u &= 2\pi i \left\{ \frac{1}{12}(2t_1 + t_2) e_{\bar{\nu}_0^*} + \frac{1}{2}t_2 e_{\bar{\nu}_{d+3}^*} \right\} \\ &= \tilde{c}_1 \tilde{t}_1 e_{\bar{\nu}_0^*} + \tilde{c}_2 \tilde{t}_2 e_{\bar{\nu}_{d+3}^*} \end{aligned}$$

by introducing some constants  $\tilde{c}_1$  and  $\tilde{c}_2$ . In short, we get the correspondence between the mirror maps  $t_1, t_2$  and the variables  $\tilde{t}_1, \tilde{t}_2$  associated with cohomology basis in  $H^{1,1}((\mathbf{C})^{d+1} \setminus Z_{f_\Delta}, \mathbf{Z})$ ,

$$\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \frac{1}{2\pi i} \begin{pmatrix} 6\tilde{c}_1 & -\tilde{c}_2 \\ 0 & 2\tilde{c}_2 \end{pmatrix} \begin{pmatrix} \tilde{t}_1 \\ \tilde{t}_2 \end{pmatrix} .$$

We put the conjecture that the series expansion of the correlation functions with respect to the parameters  $q_j = \exp(2\pi i \tilde{t}_j)$  ( $j = 1, 2$ ) should have the integral coefficients. Considering the asymptotic behaviours of  $\tilde{t}_1, \tilde{t}_2$ ,

$$\begin{aligned} 2(d+1)\tilde{c}_1\tilde{t}_1 &= 2\pi i(2t_1 + t_2) \approx 2 \log \frac{1}{(2(d+1)\psi)^{d+1}} , \\ 2\tilde{c}_2\tilde{t}_2 &= 2\pi i t_2 \approx -2 \log(2\phi) , \end{aligned}$$

we may choose  $\tilde{c}_1 = 1/(d+1)$ ,  $\tilde{c}_2 = -1$  naturally. Then we obtain the relations,

$$\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \frac{1}{2\pi i} \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} \tilde{t}_1 \\ \tilde{t}_2 \end{pmatrix} ,$$

or

$$\begin{pmatrix} \tilde{t}_1 \\ \tilde{t}_2 \end{pmatrix} = 2\pi i \begin{pmatrix} 1 & 1/2 \\ 0 & -1/2 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} ,$$

and determine the Kähler cone  $\sigma(K)$  as

$$\sigma(K) = \left\{ \tilde{t}_1 h_1 + \tilde{t}_2 h_2 ; \tilde{t}_1 + \tilde{t}_2 > 0, \tilde{t}_2 < 0 \right\}$$

with

$$h_1 = \frac{1}{d+1} e_{\bar{\nu}_0^*} , \quad h_2 = -e_{\bar{\nu}_{d+3}^*} .$$

Collecting all the above results, we can define the  $d$  point correlation functions associated with the  $\tilde{t}_1, \tilde{t}_2$ ,

$$\kappa_{\tilde{t}_{i_1} \tilde{t}_{i_2} \cdots \tilde{t}_{i_d}} = \frac{1}{(\varpi_0)^2} \sum_{j'_1 j'_2 \cdots j'_d} \frac{\partial w_{j'_1}}{\partial \tilde{t}_{i_1}} \cdot \frac{\partial w_{j'_2}}{\partial \tilde{t}_{i_2}} \cdots \frac{\partial w_{j'_{d-1}}}{\partial \tilde{t}_{i_{d-1}}} \cdot \frac{\partial w_{j'_d}}{\partial \tilde{t}_{i_d}} \cdot \kappa_{w_{j'_1} w_{j'_2} \cdots w_{j'_{d-1}} w_{j'_d}} ,$$

where  $w_i = \{\bar{x}, \bar{y}\}$ . The couplings  $\kappa_{w_{j'_1} w_{j'_2} \dots w_{j'_{d-1}} w_{j'_d}}$  are symmetric completely under the permutations of any two indices and defined as,

$$\kappa_{\bar{x}\bar{x}\dots\bar{x}\bar{x}} := W_{d,0} ,$$

$$\kappa_{\bar{x}\bar{x}\dots\bar{x}\bar{y}} := W_{d-1,1} ,$$

$$\kappa_{\bar{x}\bar{x}\dots\bar{y}\bar{y}} := W_{d-2,2} ,$$

...

$$\kappa_{\bar{x}\bar{x}\bar{y}\dots\bar{y}} := W_{2,d-2} ,$$

$$\kappa_{\bar{x}\bar{y}\dots\bar{y}\bar{y}} := W_{1,d-1} ,$$

$$\kappa_{\bar{y}\bar{y}\dots\bar{y}\bar{y}} := W_{0,d} .$$

There are useful equations one can derive from the integral representation of the mirror maps (41,43) in section 3.3.

$$\begin{aligned} t \equiv \tilde{t}_1 &= -\frac{1}{4} + \frac{1}{8\varpi_0} \int_C \frac{d\nu}{(\sin \pi\nu)^2} \frac{\Gamma((d+1)\nu+1)}{[\Gamma(\nu+1)]^{d+1}} , (2(d+1)\psi)^{-(d+1)\nu} \\ &\quad \times [u_\nu(\phi) \cos \pi\nu + u_\nu(-\phi)] , \end{aligned} \quad (52)$$

$$\begin{aligned} s \equiv \tilde{t}_2 &= -\frac{1}{4} + \frac{1}{8\varpi_0} \int_C \frac{d\nu}{(\sin \pi\nu)^2} \frac{\Gamma((d+1)\nu+1)}{[\Gamma(\nu+1)]^{d+1}} (2(d+1)\psi)^{-(d+1)\nu} \\ &\quad \times [u_\nu(\phi) \cos \pi\nu - u_\nu(-\phi)] . \end{aligned} \quad (53)$$

It follows that,

$$\begin{aligned} \frac{-1}{(2(d+1)\psi)^{d+1}} &= q \exp \left\{ -\frac{1}{\varpi_0} \sum_{n=1}^{\infty} \sum_{r=0}^{[n/2]} \frac{((d+1)n)!}{(n!)^d} \left\{ \frac{-1}{(2(d+1)\psi)^{d+1}} \right\}^n \frac{(2\phi)^{n-2r}}{(r!)^2(n-2r)!} \right. \\ &\quad \left. \times \{(d+1)\Psi((d+1)n+1) - d\Psi(n+1) - \Psi(r+1)\} \right\} , \end{aligned} \quad (54)$$

$$\phi = \cosh \left\{ s + \frac{\sqrt{\phi^2 - 1}}{2\varpi_0} \sum_{n=1}^{\infty} \frac{((d+1)n)!}{(n!)^{d+1}} \left\{ \frac{-1}{(2(d+1)\psi)^{d+1}} \right\}^n \tilde{f}_n \right\} , \quad (55)$$

$$q := e^t ,$$

where the progression  $\tilde{f}_n$  is defined by the recurrence relation,

$$n\tilde{f}_n = 2(2n-1)\phi\tilde{f}_{n-1} - 4(n-1)(\phi^2 - 1)\tilde{f}_{n-2} \quad (n \geq 2) ,$$

$$\text{with } \tilde{f}_0 = 0, \tilde{f}_1 = 4 .$$

We can solve the simultaneous equation (54,55) iteratively and can express  $\frac{-1}{(2(d+1)\psi)^{d+1}}$  and  $\phi$  as functions with respect to the variables  $q$  and  $s$ . For simplicity we restrict dimension 5

case.

$$\begin{aligned}
& \frac{-1}{(12\psi)^6} \\
&= q - 13968 q^2 \cosh(s) \\
&\quad + q^3 (-151472592 - 20113272 \cosh(2s)) \\
&\quad + q^4 (-10207462523904 \cosh(s) - 575186850048 \cosh(3s)) \\
&\quad + q^5 (-470447935319475000 - 351288730023768000 \cosh(2s) \\
&\quad - 11245321006807500 \cosh(4s)) \\
&\quad + q^6 (-63496742982637936184064 \cosh(s) - 12377026715700759881472 \cosh(3s) \\
&\quad - 257793953453030443392 \cosh(5s)) \dots , \\
\phi &= \cosh(s) + q (-720 + 720 \cosh(2s)) \\
&\quad + q^2 (-6457320 \cosh(s) + 6457320 \cosh(3s)) \\
&\quad + q^3 (-331383545280 + 228742377600 \cosh(2s) + 102641167680 \cosh(4s)) \\
&\quad + q^4 (-18870126574818240 \cosh(s) + 16782401580155820 \cosh(3s) \\
&\quad + 2087724994662420 \cosh(5s)) \\
&\quad + q^5 (-1229739557314841424000 + 372633864619868532000 \cosh(2s) \\
&\quad + 808367768482328657856 \cosh(4s) + 48737924212644234144 \cosh(6s)) \dots .
\end{aligned}$$

Gathering the above formulae, we can calculate the five point functions  $\kappa_{\bar{t}t\bar{t}t\bar{t}}$ ,

$$\begin{aligned}
\kappa_{tttt} &= 1 + 113904 q \cosh(s) \\
&\quad + q^2 (12257897184 + 5810000400 \cosh(2s)) \\
&\quad + q^3 (2684392242065856 \cosh(s) + 273249514754496 \cosh(3s)) \\
&\quad + q^4 (257981829874371295200 + 220117176334492091136 \cosh(2s) \\
&\quad + 12304491429473532432 \cosh(4s)) \\
&\quad + q^5 (65961563983029997185594240 \cosh(s) \\
&\quad + 15336955823470891838722080 \cosh(3s) \\
&\quad + 538880531227754226547104 \cosh(5s)) + \dots ,
\end{aligned}$$

$$\begin{aligned}
\kappa_{ttts} &= 73584 q \operatorname{Sinh}(s) + 4431056400 q^2 \operatorname{Sinh}(2s)
\end{aligned}$$

$$\begin{aligned}
& + q^3 (740692484764992 \operatorname{Sinh}(s) + 223867206553536 \operatorname{Sinh}(3s)) \\
& + q^4 (94763126006876550528 \operatorname{Sinh}(2s) \\
& \quad + 10497751479836350992 \operatorname{Sinh}(4s)) \\
& + q^5 (11679174337213718521430400 \operatorname{Sinh}(s) \\
& \quad + 8125530240559872426667488 \operatorname{Sinh}(3s) \\
& \quad + 472005886382041853287584 \operatorname{Sinh}(5s)) + \dots ,
\end{aligned}$$

$\kappa_{ttss}$

$$\begin{aligned}
& = -1 + 33264q \cosh(s) \\
& + q^2 (-2091181536 + 3052112400 \cosh(2s)) \\
& + q^3 (-97568302413888 \cosh(s) + 174484898352576 \cosh(3s)) \\
& + q^4 (-24898684426198921440 \\
& \quad + 23835103083824697216 \cosh(2s) \\
& \quad + 8691011530199169552 \cosh(4s)) \\
& + q^5 (-3002113033926710657136768 \cosh(s) \\
& \quad + 3450494265438241725942816 \cosh(3s) \\
& \quad + 405131241536329480028064 \cosh(5s)) + \dots ,
\end{aligned}$$

$\kappa_{tsss}$

$$\begin{aligned}
& = -2 \coth(s) + q (47880 \operatorname{Csch}(s) - 3528 \cosh(2s) \operatorname{Csch}(s)) \\
& + q^2 (690544440 \coth(s) + 836584200 \cosh(3s) \operatorname{Csch}(s)) \\
& + q^3 (199939297711200 \operatorname{Csch}(s) \\
& \quad - 142093529019456 \cosh(2s) \operatorname{Csch}(s) \\
& \quad + 62551295075808 \cosh(4s) \operatorname{Csch}(s)) \\
& + q^4 (14369096305624811616 \coth(s) \\
& \quad - 5913414261982747944 \cosh(3s) \operatorname{Csch}(s) \\
& \quad + 3442135790280994056 \cosh(5s) \operatorname{Csch}(s)) \\
& + q^5 (2001718581208029475806528 \operatorname{Csch}(s) \\
& \quad - 1099251328808483969708592 \cosh(2s) \operatorname{Csch}(s) \\
& \quad + 256067578543639354918560 \cosh(4s) \operatorname{Csch}(s) \\
& \quad + 169128298345308553384272 \cosh(6s) \operatorname{Csch}(s)) + \dots ,
\end{aligned}$$

$$\mathcal{K}_{tssss}$$

$$\begin{aligned}
&= -3 - 4 \operatorname{Csch}(s)^2 + q (56196 \operatorname{coth}(s) \operatorname{Csch}(s) - 11844 \cosh(3s) \operatorname{Csch}(s)^2) \\
&+ q^2 (-1352281068 \operatorname{Csch}(s)^2 + 3051741096 \cosh(2s) \operatorname{Csch}(s)^2 \\
&\quad + 73556100 \cosh(4s) \operatorname{Csch}(s)^2) \\
&+ q^3 (102123927100416 \operatorname{coth}(s) \operatorname{Csch}(s) \\
&\quad + 19002269006736 \cosh(3s) \operatorname{Csch}(s)^2 \\
&\quad + 18930070487664 \cosh(5s) \operatorname{Csch}(s)^2) \\
&+ q^4 (-5814119886687093744 \operatorname{Csch}(s)^2 \\
&\quad + 21888592432000212852 \cosh(2s) \operatorname{Csch}(s)^2 \\
&\quad - 3500145461349429192 \cosh(4s) \operatorname{Csch}(s)^2 \\
&\quad + 1269382907731201668 \cosh(6s) \operatorname{Csch}(s)^2) \\
&+ q^5 (592313888205518158535976 \operatorname{coth}(s) \operatorname{Csch}(s) \\
&\quad + 1052832649994450804745144 \cosh(3s) \operatorname{Csch}(s)^2 \\
&\quad - 169749335609769672076104 \cosh(5s) \operatorname{Csch}(s)^2 \\
&\quad + 67845487961226183377256 \cosh(7s) \operatorname{Csch}(s)^2) + \dots ,
\end{aligned}$$

$$\mathcal{K}_{sssss}$$

$$\begin{aligned}
&= -4 \operatorname{coth}(s) - 8 \operatorname{coth}(s) \operatorname{Csch}(s)^2 \\
&+ q (-73080 \operatorname{Csch}(s) - 43848 \cosh(2s) \operatorname{Csch}(s)) \\
&+ q^2 (8605688184 \operatorname{coth}(s) - 542359800 \cosh(3s) \operatorname{Csch}(s)) \\
&+ q^3 (37219523344224 \operatorname{Csch}(s) \\
&\quad + 353500849433280 \cosh(2s) \operatorname{Csch}(s) \\
&\quad + 13168986874848 \cosh(4s) \operatorname{Csch}(s)) \\
&+ q^4 (29797773008663870208 \operatorname{coth}(s) \\
&\quad + 5789880986681623032 \cosh(3s) \operatorname{Csch}(s) \\
&\quad + 1635395840643812616 \cosh(5s) \operatorname{Csch}(s)) \\
&+ q^5 (847125331875900500330304 \operatorname{Csch}(s) \\
&\quad + 3137482755359009546571216 \cosh(2s) \operatorname{Csch}(s) \\
&\quad - 87672381735274833962592 \cosh(4s) \operatorname{Csch}(s) \\
&\quad + 102253653499596180124752 \cosh(6s) \operatorname{Csch}(s)) + \dots ,
\end{aligned}$$

where  $t \equiv \tilde{t}_1$  and  $s \equiv \tilde{t}_2$ . The classical parts of these five point correlation functions can be interpreted as the intersections of some divisors. Firstly we consider a divisor  $L$ , which is defined by a surface of degree  $2(d+1)$  in  $\mathbf{P}_{d+1}[2, 2, 2, \dots, 2, 2, 1](2(d+1))$  with one parameter  $\lambda$ ,

$$z_1^{d+1} + z_2^{d+1} + \dots + z_{d-1}^{d+1} + z_d^{d+1} + (1 + \lambda)z_{d+1}^{2(d+1)} = 0 .$$

Because any two distinct elements of the linear system  $|L|$  are disjoint, we obtain  $L \cdot L = 0$ . Secondly we take a linear system  $|H|$ , which is generated by degree 2 polynomials. This linear system is written as

$$|H| = |2L + E| ,$$

where  $E$  is the exceptional divisor. Because the variables  $z_1, z_2, \dots, z_{d-1}, z_d$  are degree 2, we express these  $d$  variables as some quadratic homogeneous polynomials in  $z_{d+1}$  and  $z_{d+2}$  in the calculation of an intersection  $H^d$ . In such an calculation, we have a homogeneous polynomial of degree  $2(d+1)$  in the variables  $z_{d+1}$  and  $z_{d+2}$  and we get  $H^d = 2(d+1)$ . By using the similar argument, one can read  $H^{d-1} \cdot L = d+1$ . Using the relations,

$$\begin{aligned} H^d &= 2(d+1) , \quad H^{d-1} \cdot L = d+1 , \quad L^2 = 0 , \\ H &= 2L + E , \end{aligned}$$

we can calculate intersection numbers of the divisors  $H$  and  $L$ ,

$$\begin{aligned} H^d &= 2(d+1) , \quad H^{d-1} \cdot E = 0 , \quad H^{d-2} \cdot E^2 = -2(d+1) , \dots \\ H^2 \cdot E^{d-2} &= -2(d+1)(d-3) , \quad H \cdot E^{d-1} = -2(d+1)(d-2) , \quad E^d = -2(d+1)(d-1) . \end{aligned}$$

Note the following equations,

$$\begin{aligned} -2 \coth(s) &= -2 - 4 \cdot \frac{p^{-2}}{1 - p^{-2}} , \\ -3 - 4 \operatorname{Csch}(s)^2 &= -3 - 16 \cdot \frac{p^{-2}}{(1 - p^{-2})^2} , \\ -4 \coth(s) - 8 \coth(s) \operatorname{Csch}(s)^2 &= -4 - \frac{40p^{-2}}{(1 - p^{-2})^2} - \frac{56p^{-4}}{(1 - p^{-2})^3} - \frac{8p^{-6}}{(1 - p^{-2})^3} , \\ p &:= e^s . \end{aligned}$$

we obtain the classical parts of the five point correlation functions in the 5 dimensinal case,

$$\begin{aligned} \kappa_{tttt}^0 &= 1 , \quad \kappa_{ttts}^0 = 0 , \quad \kappa_{ttss}^0 = -1 , \\ \kappa_{tsss}^0 &= -2 , \quad \kappa_{tsss}^0 = -3 , \quad \kappa_{ssss}^0 = -4 . \end{aligned}$$

These numbers coincide with the intersection numbers up to a common overall factor 12. As for the corrections in the couplings, these are not understood in the geometrical terms yet.

## 5 The monodromy

In this section, we study monodromy transformations and give monodromy matrices.

### 5.1 The monodromy

The defining equation of two parameter family of the Calabi-Yau  $d$ -fold was represented in (29). If the conditions  $\partial p/\partial z_i = 0$ , ( $i = 1, 2, \dots, d+2$ ) are satisfied for some  $\psi, \phi$ , this variety gets singular. These conditions are rewritten in the following four cases,

1. Along the locus  $\phi + 2^d \psi^{d+1} = \pm 1$ , the  $d$ -fold  $\{p = 0\}/G$  has a collection of conifold points. ( $C_{con}$ )
2. Along the locus  $\phi = \pm 1$ , the  $d$ -fold  $\{p = 0\}/G$  has isolated singularities. ( $C_1$ )
3. Along the locus  $\psi = 0$ , the  $d$ -fold  $\{p = 0\}/G$  gets singular. ( $C_0$ )
4. When the parameters  $\phi$  and  $\psi$  tend to infinity, one gets a singular fold ( $C_\infty$ ),

$$\left\{ 2(d+1)\psi z_1 z_2 \cdots z_d z_{d+1} z_{d+2} + 2\phi z_{d+1}^{d+1} z_{d+2}^{d+1} = 0 \right\} /G .$$

The group  $G$  is characterized by its elements  $g \in G$ ,

$$g = (\alpha^{2a_1}, \alpha^{2a_2}, \dots, \alpha^{2a_{d-1}}, \alpha^{2a_d}, \alpha^{a_{d+1}}, \alpha^{a_{d+2}}) , \quad \alpha := \exp \frac{2\pi i}{2(d+1)} ,$$

acting on  $(z_1, z_2, \dots, z_{d-1}, z_d, z_{d+1}, z_{d+2}, \psi, \phi)$  as

$$(\alpha^{2a_1} z_1, \alpha^{2a_2} z_2, \dots, \alpha^{2a_{d-1}} z_{d-1}, \alpha^{2a_d} z_d, \alpha^{a_{d+1}} z_{d+1}, \alpha^{a_{d+2}} z_{d+2}, \alpha^{-a} \psi, \alpha^{-(d+1)a} \phi) ,$$

with  $a := 2a_1 + 2a_2 + \dots + 2a_{d-1} + 2a_d + a_{d+1} + a_{d+2}$ . This group  $G$  acts on the parameter space  $\{(\psi, \phi)\}$  as

$$(\psi, \phi) \rightarrow (\alpha\psi, -\phi) . \quad (56)$$

If one rotates the values of the parameters  $(\psi, \phi)$  around the special values defined in 1.,2.,3.,4. analytically, homology cycles  $\gamma_i$  of the variety  $\{p = 0\}/G$  turn into linear combinations of other homology cycles  $\sum_j c_{ij} \gamma_j$ . As a consequence, the periods also change into linear combinations of others. The matrices associated with this linear transformations are called monodromy matrices and contain geometrical information. Recall the discriminant loci  $\Delta_1, \Delta_2, \Delta_3$  and  $\Delta_4$  in section 4.1. These can be rewritten as

$$\Delta_1 = (1 - \bar{x})^2 - \bar{x}^2 \bar{y} = \frac{1}{(2^d \psi^{d+1})^2} (2^d \psi^{d+1} + \phi + 1) (2^d \psi^{d+1} + \phi - 1) ,$$

$$\begin{aligned}\Delta_2 &= 1 - \bar{y} = \frac{1}{\phi^2} (\phi + 1)(\phi - 1) , \\ \Delta_3 &= \bar{x} = -\frac{\phi}{2^d \psi^{d+1}} , \\ \Delta_4 &= \bar{y} = \frac{1}{\phi^2} .\end{aligned}$$

The zero loci of  $\Delta_1$  are  $\phi + 2^d \psi^{d+1} = \pm 1$  and associate with the case 1. Secondly the zero loci of  $\Delta_2$  are  $\phi = \pm 1$  and correspond to the case 2. Thirdly the discriminant  $\Delta_3 = \bar{x}$  is the fixed point of the transformation (56). Lastly the discriminant  $\Delta_4$  have zero at  $\phi = \infty$  and corresponds to the case 4.

## 5.2 Monodromy matrices

In this subsection, we treat the monodromy matrices of the fivefold. In order to obtain the monodromy matrices, let us take a basis  $v$  of the periods,

$$v := {}^t(\varpi_0, \varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5, \varpi_6, \varpi_7, \varpi_8, \varpi_9) .$$

Firstly we consider the conifold singularity along the locus  $\phi + 2^5 \psi^6 = 1$ . By means of the same argument made in [9], it can be understood that the periods have the structure,

$$\varpi_j(\psi, \phi) = \frac{c_j}{2\pi i} (\varpi_1 - \varpi_0) \log(2^5 \psi^6 - \phi - 1) + f_j(\psi, \phi) ,$$

where the  $f_j$  is an analytic functions in the neighbour of this singular point and the  $c_j$  are some constant coefficients. The monodromy transformation  $\mathcal{T}$  of this singularity acts on the periods,

$$\mathcal{T} \varpi_j = \varpi_j + c_j (\varpi_1 - \varpi_0) .$$

We obtain the coefficients  $c_j$ ,

$$\begin{aligned}c_0 &= -1 , c_1 = -1 , c_2 = 1 , c_3 = 5 , c_4 = -5 , \\ c_5 &= -10 , c_6 = 10 , c_7 = 10 , c_8 = -10 , c_9 = -5 .\end{aligned}$$

The monodromy matrix  $T$  associated with  $\mathcal{T}$  are found,

$$T = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -5 & 5 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & -5 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 10 & -10 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -10 & 10 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -10 & 10 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 10 & -10 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 5 & -5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Secondly let us consider the isolated singularity along the locus  $\phi = 1$ . By the monodromy transformation  $\mathcal{B}$  of this singularity, the function  $u_\nu(\phi), u_\nu(-\phi)$  defined in (31) are transformed linearly,

$$\begin{aligned} u_\nu(\phi) &\rightarrow (1 - e^{2\pi i \nu}) u_\nu(\phi) + e^{\pi i \nu} u_\nu(-\phi) , \\ u_\nu(-\phi) &\rightarrow e^{\pi i \nu} u_\nu(\phi) . \end{aligned}$$

Using the integral representation of the periods with the change of the contour  $C$  to enclose the poles  $\nu = 0, 1, 2, \dots$  clockwise, we obtain the monodromy matrix  $B$  associated with  $\mathcal{B}$ ,

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 5 & -5 & 6 & -5 & 5 & -5 & 5 & -5 & 5 & -5 \\ -5 & 5 & -6 & 6 & -4 & 5 & -5 & 5 & -5 & 5 \\ -10 & 10 & -10 & 10 & -9 & 10 & -10 & 10 & -10 & 10 \\ 10 & -10 & 10 & -10 & 9 & -9 & 11 & -10 & 10 & -10 \\ 10 & -10 & 10 & -10 & 10 & -10 & 11 & -10 & 10 & -10 \\ -10 & 10 & -10 & 10 & -10 & 10 & -11 & 11 & -9 & 10 \\ -5 & 5 & -5 & 5 & -5 & 5 & -5 & 5 & -4 & 5 \end{pmatrix}.$$

Thirdly the monodromy matrix  $A$  associated with the monodromy transformation  $\mathcal{A}$  along the locus  $\psi = 0$ , i.e.  $(\psi, \phi) \rightarrow (\alpha\psi, -\phi)$  can be obtained easily,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \end{pmatrix}.$$

Lastly the monodromy transformation at the infinity is expressed  $(\mathcal{BTA})^{-1}$ . Thus the monodromy matrix can be read,

$$(ATB)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 2 & 1 \end{pmatrix}.$$

## 6 Conclusion

In this paper, we constructed a two parameter family of the Calabi-Yau d-fold by the method of the toric variety. The correlation functions associated with the complex moduli in the mirror manifold are meromorphic functions and become singular when at least one of the discriminant loci  $\Delta_1, \Delta_2, \Delta_3, \Delta_4$  vanishes. These discriminant loci correspond to the singular points of the manifold in section 5 and these singularities are reflected in the correlation functions associated with the complex moduli space. By contrast, the correlation functions associated with the Kähler moduli in the original manifold have quantum corrections. The classical parts of these Kähler couplings can be interpreted in geometrical terms. Quantum corrections in these couplings in d-fold have more complicated form than threefolds. In the case of Calabi-Yau threefolds, there are isolated rational curves in these threefolds and quantum corrections in the three point couplings associated with the Kähler moduli can be interpreted as the numbers of these curves. As already indicated for the one parameter models [24], there are families of rational curves in higher ( $\geq 4$ ) dimensions. So our results seem to reflect families of instantons with continuous parameters more manifestly than cases of the one parameter models and it is expected to be interpreted as intersection numbers of some Chern class with Kähler forms.

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